$$=\frac{8r^2}{\pi^2}\int_0^{\frac{1}{2}\pi}(\tfrac{1}{8}\pi^2+\tfrac{1}{2}\theta^2-\tfrac{1}{2}\pi\theta+\tfrac{1}{2}\cos^2\theta-\tfrac{1}{2}\pi\sin\theta\cos\theta+\theta\sin\theta\cos\theta)d\theta=\tfrac{1}{6}\pi r^2.$$

Second Case. The two chords AB and $A'B'$ may be on *different* sides of the diameter XY; that is, the chord $A'B'$ now becomes the chord MN in position.

$\therefore$ Area $AMNB=U_2=(\phi+\theta+\sin\phi\cos\phi+\sin\theta\cos\theta)r^2$; and consequently,

$$A_2=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}U_2 d\theta\, d\phi\, d\omega\, d\psi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}d\theta\, d\phi\, d\omega\, d\psi}$$

$$=\frac{4r^2}{\pi^2}\int_0^{\frac{1}{2}\pi}(\tfrac{1}{8}\pi^2+\tfrac{1}{2}\pi\theta+\tfrac{1}{2}+\tfrac{1}{2}\pi\sin\theta\cos\theta)d\theta=\tfrac{1}{2}(\pi+1/\pi)r^2.$$

$\therefore A=\frac{1}{2}(A_1+A_2)=(\frac{1}{3}\pi+1/4\pi)r^2$, which is the required average area.

Corollary. If $r=1$, $A=\frac{113}{314}$; that is, the required average is slightly greater than one-third of the given circle.

CALCULUS.

201. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve $\int\int_0^{dy/dx}\frac{dw}{1+w^2}=0.$

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\int\int_0^{dy/dx}\frac{dw}{1+w^2}=0.\qquad \int\Big[\tan^{-1}w\Big]_0^{dy/dx}=0.\qquad \therefore\int\tan^{-1}\frac{dy}{dx}=0.\qquad \therefore\frac{dy}{dx}=0.$$

$y=C=$a constant.

202. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the complete primitive of $y=2px+ap^2$. Regard the primitive as the equation giving the arbitrary constant, and if the primitive has equal roots discuss the equation expressing that condition.

Solution by G. W. GREENWOOD, M. A.

Differentiating with regard to x we obtain an equation which may be written

$$p\frac{dx}{dp}+2x=-2ap.$$

Integrating, we have $3p^2x+2ap^3=c$, where c is a constant. Eliminating p between this equation and the original equation, we get

$$a^2c^2+2c(2x^3+3axy)-y^2(4ay+3x^2)=0.$$

This is the primitive equation. If the values of c are equal, then

$$(2x^3+3axy)^2+a^2y^2(4ay+3x^2)=0;\ i.\ e.,\ (x^2+ay)^3=0.$$

When the discriminating equation has a factor cubed, this factor equated to zero gives the cuspidal locus of the given family of curves. In fact, in the present case there is a cusp at the point ($a^{\frac{2}{3}}c^{\frac{1}{3}}$, $-$ $a^{\frac{1}{3}}c^{\frac{2}{3}}$).

GEOMETRY.

260. **Proposed by W. J. GREENSTREET, M. A., Stroud, England.**

Perpendiculars to the radius vector are drawn through points on $r=a+b\cos n\theta$. Find the radius of curvature of their envelope at a point at a given distance from the origin.

I. **Solution by G. B. M. ZERR, A. M., Ph. D.**

$r=a+b\cos n\theta=R\cos(\phi-\theta)=p$..........(1).

Differentiating (1) we get, $-bn\sin n\theta=R\sin(\phi-\theta)$..........(2).

$\therefore R^2=(a+b\cos n\theta)^2+b^2n^2\sin^2 n\theta$..........(3).

$\therefore R^2=p^2+b^2n^2-n^2(p-a)^2$..........(4).

(3) and (4) are both equations to the envelope.

Radius of curvature$=\rho=R(dR/dp)$.

$\therefore \rho=p+an^2-n^2p=r+an^2-n^2r=an^2+(1-n^2)(a+b\cos n\theta)$
$=a+b\cos n\theta-bn^2\cos n\theta=a-b(n^2-1)\cos n\theta$.

II. **Solution by G. W. GREENWOOD. M. A.**

Let $(p,\ \alpha)$ be the foot of the perpendicular from the origin upon the tangent at a point P of the envelope. Then at P,

$$\rho=p+d^2p/d\alpha^2=a+b\cos n\alpha-bn^2\cos n\alpha,$$

since $\quad p=a+b\cos n\alpha$.

MECHANICS.

182. **Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.**

I have a tank, the lower part of which is a hemisphere 22 feet in diameter. The rest is a cylinder 22 feet in diameter, and altitude 28 feet. This tank is connected with the earth by a vertical stand-pipe 10 inches in diameter, 130 feet long, extending 2 feet into the tank. The tank is filled by a $2\frac{1}{2}$ inch pipe 65 feet long, having one right-angled elbow delivering the water into the bottom of the stand-pipe from a steam pump under 96 pounds gauge pressure. How long will it take to fill the pipe?